Combinatorial Networks, 2015 Spring
Homework 1

1. Let n, r be be positive integers and $n \geq r$. Give a **combinatorial proof** of

$$\binom{2n}{2r} \equiv \binom{n}{r} (mod\ 2).$$

2. For any integer $n \geq 2$, let $\pi(n)$ be the number of primes in $\{1, 2, ..., n\}$.

(a) Prove that the product of all primes p satisfying $m < p \leq 2m$ is at most $\binom{2m}{m}$, where $m \geq 1$ is any integer.

(b) Use (a) to prove that $\pi(n) \leq \frac{Cn}{\log n}$ for some absolute constant C. (Hint: by induction and use the estimation on $\binom{2m}{m}$)

3. How many ways are there to seat n couples at a round table with $2n$ chairs in such a way that none of the couples sit next to each other? If one seating plan can be obtained from other plan by a rotation, then we will view them as one plan.

4. Given a tree T on vertices $1, 2, \ldots, n$. Let $A_1, A_2, ..., A_n$ be n arbitrary subsets of a ground set Ω (such that each subset A_i is corresponding to the vertex i). Prove that

$$|A_1 \cup A_2 \cup ... \cup A_n| \leq \sum_{i=1}^{n} |A_i| - \sum_{ij \in E(T)} |A_i \cap A_j|.$$

Here, $ij \in E(T)$ expresses an edge of the tree T jointing vertex i and vertex j.

5. Let $D(n)$ be the number of permutations π of $[n]$ such that $\pi(i) \neq i$ for any $i \in [n]$. Prove that $D(n+1) = n[D(n-1) + D(n)]$ for all $n \geq 2$.

Note: Using double counting; a proof by plugging in the precise formula of $D(n)$ is not accepted.

6. Find a sequence $a_0, a_1, a_2, ...$ such that for all integer $n \geq 0$

$$\sum_{k=0}^{n} a_k a_{n-k} = \binom{n+2}{2}.$$

7. Let S_n be the set of strings of length n with entries from the set $\{a, b\}$ and with **NO** "abb" occurring (in three consecutive positions). Let $s_n = |S_n|$, where $s_0 = 1$, $s_1 = 2$ and $s_2 = 4$. Prove that for any integer $n \geq 3$,

$$s_n = s_{n-1} + s_{n-2} + 1.$$

8. Find the maximum number of line segments that a Hasse diagram of poset $(X, \prec)$ with $|X| = n$ can have. Then define a poset which achieves this maximum number and draw its Hasse diagram.

9. For any point $p \in R^d$ in d-dimension, write $p = (p_1, p_2, ..., p_d)$. A set $\mathcal{P}$ of points in R^d is called *good*, if for each $i \in [d]$, the i^{th} coordinates of these points are distinct. Given two points $p, q \in R^d$, define $box(p, q) := \{x \in R^d : \min\{p_i, q_i\} \leq x_i \leq \max\{p_i, q_i\}$ for each $i\}$ as the box determined by points p, q.

Prove that in any good set $\mathcal{P}$ of $2^{2^{d-1}} + 1$ points of R^d, there is a point $x \in \mathcal{P}$ which is in the box determined by two of the other points in $\mathcal{P}$.

10.(1). For any integers $k, l \geq 1$, construct a sequence of kl distinct integers with no increasing subsequence of length $k+1$ and with no decreasing subsequence of length $l+1$.
(2). Construct an explicit 2-edge-coloring of complete graph on k^2 vertices to show that $R(k+1, k+1) \geq k^2 + 1$.

11. Use Ramsey's theorem to prove: for every integer $k \geq 2$, there is an integer n such that every sequence of n distinct real numbers contains a monotone subsequence of k real numbers.

(You must use Ramsey's theorem and cannot use the Erdős-Szekeres theorem.)

12. Let $s, t, n \geq 2$ be integers. Prove that if

$$\binom{n}{s} \cdot 2^{-\binom{s}{2}} + \binom{n}{t} \cdot 2^{-\binom{t}{2}} < 1,$$

then Ramsey number $R(s, t) > n$.

13. For all integers $r \geq 3$ and $s, t \geq r$, prove that hypergraph Ramsey number

$$R^{(r)}(s, t) \leq 2^{\binom{R^{(r-1)}(s-1, t-1)}{r-1}}.$$

(In class, we proved the case $r = 3$.)

14. Prove the following fact for graph Ramsey number on k-colors

$$2^k \leq R_k(3, 3, ..., 3) \leq (k+1)!$$

15. Let $n > 0$ be an even integer. Let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets of $[n]$ such that $\mathcal{F}$ contains no three distinct sets A, B, C satisfying $A \subset B \subset C$. Show that $|\mathcal{F}| \leq 2\binom{n}{n/2}$.

16. Let X be an n-element set and let $S_1, S_2, ..., S_n$ be subsets of X such that $|S_i \cap S_j| \leq 1$ for any $i \neq j$. Prove that there exists some subset S_i with $|S_i| \leq C\sqrt{n}$, for absolute constant C (independent of the choice of n).

17. Let G be a bipartite graph with bipartition (A, B), where $|A| = n, |B| = m$. Suppose that G contains no $K_{2,2}$ as a subgraph. Prove that G has at most $C \cdot (m\sqrt{n} + n)$ edges for some absolute constant C.